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Topologimeter and the Problem of Physical Interpretation of Topology Lattice

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The collection of all topologies on the set of three points is studied, treating the topology as a quantum-like observable. This turns out to be possible under the assumption of the asymmetry between the spaces of bra and ket vectors. Analogies between the introduced topologimeter and Stern–Gerlach experiments are outlined.

INTRODUCTION

An important problem in modern theoretical physics is the problem of quantum topology. What is quantum topology? Can one have something like a wave function for topologies? Can one speak about the probability of this or that topology, and is it possible to speak about a probability calculus for topologies? Are "quantum jumps" between different topologies possible? It is well known that in classical physics one can use the classical probability measure, but for molecules, atoms, elementary particles, and maybe gravity one has probability amplitude descriptions in terms of wave functions. This is connected with the fact that the lattice of properties of quantum system is nondistributive (non-Boolean) (Birkhoff and von Neumann, 1936), and one has the formalism of Hilbert space, noncommutative operators, wave functions, and Heisenberg uncertainty relations.

One can speak about quantum topology if one deals with Planckian scales for space-time when gravity must be quantized. There is another interesting possibility to speak about quantum topology when, according to Leinaas and Myrheim (1991), one interprets the Pauli principle in terms of

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a nontrivial topology for configuration space of a many-particle system, so that in the EPR experiment, when one goes from two symmetrized particles to the product after measuring local observables one has a change of topologies. Here we shall investigate the simplest case of discrete topologies. From Zapatrin (1993) and Sorkin (1991) one can see that our analysis can give some insight for the continuous case as well.

1. TOPOLOGIMETER

In this paper we shall study the properties of the topology lattice for three points, continuing the investigation in Grib and Zapatrin (1992). The striking feature of this lattice and of lattices of topologies for the number of points n greater than or equal to 3 is that they are nondistributive. This makes it impossible to have classical measure on these lattices and leads to some resemblance of them to quantum systems, but without Planck's constant. As in Grib and Zapatrin (1992), consider the work of a hypothetical "topologimeter"—an apparatus which can ask and obtain answers to the question: "What is the topology of the three-point set?" We come to the conclusion that due to nondistributivity of the lattice one obtains noncausal "quantum jumps" of topologies similar to the situation in a Stern–Gerlach experiment when complementary observables are measured. Consider the triple a, (ac), (ab) of atomic topologies [the notations are the same as in Grib and Zapatrin (1992)]:

$$a \wedge ((ac) \vee (ab)) \quad \text{is not equal to} \quad (a \wedge (ac)) \vee (a \wedge (ab)) \qquad (1)$$

Here $(ac) \vee (ab)$ for our topologimeter is also some topology defined following D'Espagnat (1976), for the case of quantum logic in the sense that if (ac) is "true," then $((ac) \vee (ab))$ is true, and if (ab) is true, then $(ac) \vee (ab)$ is true.

The Boolean-minded observer comes to the following result using the topologimeter. Imagine he sees a is true. Then, since $a = a \wedge (ac) \vee (ab))$, he will say, according to his Boolean distributive logic interpreting $\wedge$ as "and" and $\vee$ as "or," that $(ac) \vee (ab)$ is true. Then he must say that either (ac) or (ab) is true. However, the atomic topologies a, (ac), (ab) are incompatible since from the Hasse diagram we see that $a \wedge (ac) = (ac) \wedge (ab) = a \wedge (ab) = 0$. The escape from this contradiction is to assume the "noncausal jump of the topology in different moments of time." Time here plays the crucial role. If a is true for $t = t_0$, then the Boolean-minded observer will say that at some other moment $t = t_1$ the topology can noncausally "jump" into some $(ac) \vee (ab)$.

So the situation here resembles that for measuring noncommuting spin operators in the Stern–Gerlach experiment. To make this analysis closer to quantum mechanics and to obtain some "quantum topology" one can try to find something like matrix representation of our lattice of topologies to compare it with usual Pauli matrices for spin. In this paper we show that such a representation can be constructed. But in contrast to quantum mechanics, one cannot have a formulation in terms of one Hilbert space and the wave function there. One must have two spaces and our matrices are operators from one space to the other. There are both commuting and noncommuting matrices. All this shows that here we have an example of a new system, different from classical as well as from quantum mechanical cases.

Our matrix representation shows that there is a direct correspondence between noncommutativity of operators and nondistributivity of the lattice.

For nondistributive triples *a*, (*ac*), (*ab*); *b*, (*bc*), (*ab*); *c*, (*bc*), (*ac*) one has a representation in terms of noncommuting "bra operators" (matrices): *a* does not commute with (*ac*) and (*ab*), *b* with (*bc*) and (*ab*), and the same with *c*.

There are distributive triples like *a*, *b*, *c* or (*ab*), (*ac*), (*bc*). That is why comparing the work of our "topologimeter" with a Stern–Gerlach experiment one can predict the following. If the topology is fixed as "*a*" ("*a*" is true), then the prediction is that at the next moment if the question is asked, "is '(*ac*)' true?" the answer is yes—the topology will jump from "*a*" to "(*ac*))." The same is true if the question is about (*ab*). But if the question is, "if at $t = t_0$, '*a*' is true, is '*b*' true at the next moment?" the answer is no. The same prediction will hold for "*c*." The situation here resembles measuring two noncommuting spin projections for a quantum particle with spin 1. For the spin-1 case there are three commuting projectors, $S_x = 1$, $S_x = 0$, $S_x = -1$, which do not commute with $S_y = 1$, $S_y = 0$, $S_y = -1$.

Nevertheless our lattice differs from the quantum logical lattice of a spin-1 particle, as shown earlier (Grib and Zapatrin, 1992). It is easy to see from Fig. 1 that not only do (*b*), (*c*) commute with (*a*), but also the atomic (*bc*) does not commute with (*b*) and (*c*).

There is no natural orthogonality in the lattice, so if one imposes some orthogonality in the 6-dimensional space $\mathcal{H}_V$ by defining a scalar product and treating mutually commuting idempotents as orthogonal projectors, one obtains a contradictory system of equations. So, orthogonal subspaces may not correspond to projectors on any topology represented in the lattice of Fig. 1. This can be treated as some superpositions of topologies which are not observable by our topologimeter.

The most interesting aspect which makes this system different from a usual quantum microparticle is that it is "classical"—"macroscopic." There is no need for a macroscopic apparatus to measure complementary properties of some microparticle. This shows that the reason for "jumps" corresponding to wave packet collapse in quantum mechanics is not connected with any intervention of a macroscopic apparatus, but is due to the interpretation of

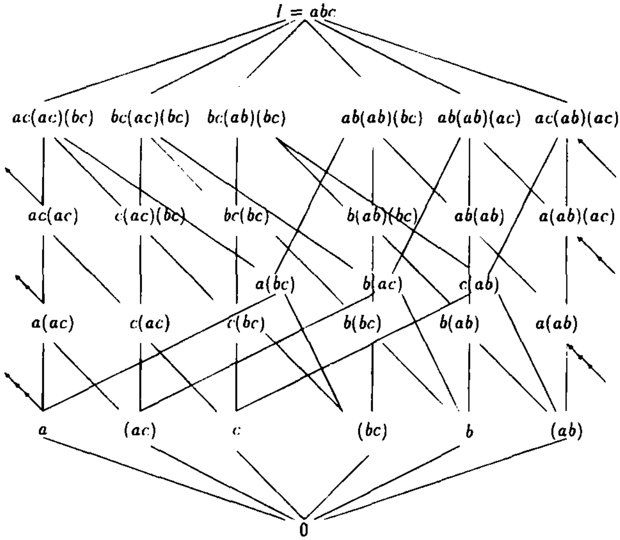

Fig. 1. The lattice $\tau(3)$.

a nondistributive lattice by some consciousness with Boolean logics. Asking a question using Boolean logics, consciousness interprets the non-Boolean structure in such a way that it uses time in order to obtain "yes–no" values for topologies as is also done in quantum logic.

But contrary to the usual Stern–Gerlach experiment for the spin-1 case, there is no need for two different complementary topologimeters for measuring noncommuting operators. It is enough to have one topologimeter, but one must look for two different moments of time in order to check values of complementary topologies.

The absence of different classical measuring apparatuses for this case leads to the absence of the necessity to have different von Neumanns measuring the Hamiltonian (von Neumann, 1955) for complementary observables. There is no "interaction" between the Boolean consciousness and the non-Boolean system in this case. This makes the investigation of the non-Boolean lattice of topologies, comparing it with usual quantum logical systems, important for understanding the problem of the role of consciousness in measurement theory in quantum physics.

2. REPRESENTATION OF THE TOPOLOGY LATTICE $\tau(3)$ BY OPERATORS

The scheme of the construction is the following. We intend to represent the elements of the topology lattice $L = \tau(3)$ by operators in a linear space.

Two 6-dimensional spaces $\mathcal{H}_V$ and $\mathcal{H}_\Lambda$ called *bra-space* and *ket-space*, respectively, are considered. The basis of $\mathcal{H}_V$ is labeled by atoms of L, and the basis of $\mathcal{H}_\Lambda$ is labeled by coatoms of L. Then each element of L has a twofold representation: as a subspace of $\mathcal{H}_V$ (called *bra representation*) or as that of $\mathcal{H}_\Lambda$ (*ket representation*). The meet operation in L is easily described in terms of the bra representation as the set-theoretic intersections of appropriate subspaces. The joins in L are associated with the set intersections in the ket representation. The object of the mathematics given below is the construction of joins in terms of $\mathcal{H}_V$ and meets in terms of $\mathcal{H}_\Lambda$.

2.1. Bra and Ket Representations

Let $L = \tau(3)$ be the lattice of all topologies on a set $X = \{a, b, c\}$ of three points. The atoms of L are the proper weakest topologies on X, each of which is associated with the only proper (that is, not equal to $\emptyset$ or X) open subset of $X = \{a, b, c\}$. Denote by $\mathcal{H}_V$ the 6-dimensional linear space with the basis labeled by these subsets:

$$\mathcal{H} = \mathrm{span}\{\mathbf{e}_{\dashv}, \mathbf{e}_{(\dashv\llcorner)}, \mathbf{e}_{\llcorner}, \mathbf{e}_{(\sqcup)}, \mathbf{e}_{\lrcorner}, \mathbf{e}_{(\dashv\lrcorner)}\}$$

The lattice $L = \tau(3)$ is a CAC (complete atomistic coatomistic) lattice (Larson and Andima, 1975), which is why each topology $\tau \in L$ is the join of atomic topologies which are weaker than τ. So τ can be unambiguously associated with the subspace $V_\tau \subseteq \mathcal{H}_V$ spanned on the appropriate basis vectors:

$$V_\tau = \mathrm{span}\{\mathbf{e}_A \mid A \in \tau\} \tag{2}$$

In other words, for any subset $A \subseteq X$, $\mathbf{e}_A \in V_\tau$ if and only if the set A is open in the topology τ.

The ket representation of L is built likewise. The coatoms of L are associated with the maximal proper topologies on $X = \{a, b, c\}$ each of which is associated with an ordered pair of elements of X (Zapatrin, 1993), denoted by $a \to b$, $c \to a$, etc. For example,

$$a \to b \quad \text{is} \quad \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}, X\}$$

$$c \to a \quad \text{is} \quad \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$$

and so on. Now introduce the 6-dimensional ket-space $\mathcal{H}_\Lambda$ with the basis labeled by ordered pairs of $X = \{a, b, c\}$:

$$\mathcal{H}_\Lambda = \mathrm{span}\{\mathbf{f}_{ab}, \mathbf{f}_{ba}, \mathbf{f}_{ac}, \mathbf{f}_{ca}, \mathbf{f}_{bc}, \mathbf{f}_{cb}\}$$

Each topology $\tau \in L$ can now be represented as the subspace $\Lambda_\tau \subseteq \mathcal{H}_\Lambda$ spanned on the basis vectors $\mathbf{f}_{xy}$, $x, y \in X$, satisfying the following condition:

$$\mathbf{f}_{xy} \in \Lambda_\tau \Leftrightarrow (\forall A \in \tau, \quad x \in A \Rightarrow y \in A) \tag{3}$$

2.2. Examples

Six examples of topologies are presented in the Table I.

2.3. Transition Between the Representations

Let $\tau \in L$ be represented by the bra-subspace $V_\tau \subseteq \mathcal{H}_V$. Denote by P_τ the projector onto V_τ. Let us construct the algorithm which builds Λ_τ by given V_τ, and vice versa. To proceed it, first introduce the *sandwich operator* S: $\mathcal{H}_V \to \mathcal{H}_\Lambda$ in matrix form as

$$S_{\lambda\nu} = \begin{cases} 0, & \nu \leq \lambda \text{ (i.e., the atom } \nu \text{ is below the coatom } \lambda) \\ 1, & \text{otherwise} \end{cases}$$

Its transpose S^T will be the operator S^T: $\mathcal{H}_\Lambda \to \mathcal{H}_V$. Consider the product SP_τ: $\mathcal{H}_V \to \mathcal{H}_\Lambda$. In the space $\mathcal{H}_\Lambda$, denote by Π_τ the projector onto the subspace Λ_τ. In particular, denote the projectors onto the basis vectors $\mathbf{e}_{xy}$, $x, y \in \{a, b, c\}$ by π_{xy}.

Lemma 1. Let $\tau \in L$ be a topology on $X = \{a, b, c\}$, V_τ be its bra representation, and P_τ be the projector in $\mathcal{H}_V$ associated with V_τ. Then the projector Π_τ in $\mathcal{H}_\Lambda$ associated with Λ_τ is

$$\Pi_\tau = \sum \{\pi_{xy} \mid \pi_{xy} SP_\tau = 0\} \tag{4}$$

Proof. π_{xy} is included in the sum (4) if and only if $S(xy, \nu) = 0$ for any atomic topology ν which is weaker than τ. Due to (3), this means that $\pi_{xy} \leq \Pi_\tau$ iff

Table I. Some Examples of Topologies

Topology τ	Bra representation	Ket representation
$a = \{\emptyset, \{a\}, X\}$ (atomic topology)	$V_a = \text{span}\{\mathbf{e}_a\}$ $\dim V_a = 1$	$\Lambda_a = \text{span}\{\mathbf{f}_{ba}\mathbf{f}_{ca}\mathbf{f}_{bc}\mathbf{f}_{cb}\}$ $\dim \Lambda_a = 4$
$= \{\emptyset, \{a, b\}, X\}$ (atomic topology)	$V_\tau = \text{span}\{\mathbf{e}_{(ab)}\}$ $\dim V_{(ab)} = 1$	$\Lambda_\tau = \text{span}\{\mathbf{f}_{ba}\mathbf{f}_{ca}\mathbf{f}_{ab}\mathbf{f}_{cb}\}$ $\dim \Lambda_{(ab)} = 4$
$a \vee b = ab(ab)$ $= \{\emptyset, a, b, \{a, b\}, X\}$	$V_\tau = \text{span}\{\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_{(ab)}\}$ $\dim V_\tau = 3 \neq \dim V_a + \dim V_b$	$\Lambda_\tau = \text{span}\{\mathbf{f}_{ca}\mathbf{f}_{cb}\}$ $\dim \Lambda_\tau = 2$
$a(bc) = \{\emptyset, a, \{b, c\}, X\}$	$V_\tau = \text{span}\{\mathbf{e}_a, \mathbf{e}_{(bc)}\}$ $\dim V_\tau = 2$	$\Lambda_\tau = \text{span}\{\mathbf{f}_{bc}\mathbf{f}_{cb}\}$ $\dim \Lambda_\tau = 2$
$(a \to b) = \{\emptyset, b, c, \{b, c\}, \{a, b\}, X\}$ (coatomic topology)	$V_\tau = \text{span}\{\mathbf{e}_b, \mathbf{e}_c, \mathbf{e}_{(ab)}, \mathbf{e}_{(bc)}\}$ $\dim V_\tau = 4$	$\Lambda_\tau = \text{span}\{\mathbf{f}_{ab}\}$ $\dim \Lambda_\tau = 4$

$$\forall \nu \nu \leq \tau \Leftrightarrow \nu \leq \lambda$$

Since the lattice L is coatomistic, $\tau = \wedge\{\lambda \mid \tau \leq \lambda\}$, thus (3) implies (4).

The following "transposed" lemma is proved likewise. Denote by p_A the projector onto the vector $\mathbf{e}_A$ in $\mathcal{H}_V$.

Lemma 2. The transition from the ket to the bra representation is described as follows:

$$P_\tau = \sum \{p_A \mid p_A S^T \Pi_\tau = 0\} \tag{5}$$

2.4. Lattice Joins in Bra Representation

Let $\sigma, \tau \in L$, and let their bra representations be $V_\sigma, V_\tau \subseteq \mathcal{H}_V$, associated with the projectors P_σ, P_τ, respectively. To build the projector $P_{\sigma\vee\tau}$, perform consecutively the transition procedures described in Lemmas 1 and 2. First form the ket representation $\Lambda_{\sigma\vee\tau}$ associated with the projector (4):

$$\Pi_{\sigma\vee\tau} = \sum \{\pi_{xy} \mid \pi_{xy} S(P_\sigma + P_\tau) = 0\} \tag{6}$$

and then go backward to $\mathcal{H}_V$. Here (6) is really the projector associated with the join since it is the meet of all upper bounds for both σ and τ (Zapatrin, 1994).

2.5. An Example

Let us explicitly build the projector associated with the join of two atomic topologies a and b. We have

$$\mathbf{e}_a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow P_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow P_b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Set up the following order of basis vectors in $\mathcal{H}_V$: $\mathbf{e}_a$, $\mathbf{e}_{ab}$, $\mathbf{e}_b$, $\mathbf{e}_{bc}$, $\mathbf{e}_c$, $\mathbf{e}_{ac}$, and of those in $\mathcal{H}_\Lambda$: $\mathbf{f}_{ca}$, $\mathbf{f}_{ba}$, $\mathbf{f}_{bc}$, $\mathbf{f}_{ac}$, $\mathbf{f}_{ab}$, $\mathbf{f}_{cb}$; then the matrix of the sandwich operator is

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Then

$$P_a + P_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$S(P_a + P_b) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the only π_{xy} satisfying (6) are the projectors onto the following vectors:

$$\mathbf{f}_{bc} = (0, 0, 1, 0, 0, 0)$$
$$\mathbf{f}_{ca} = (1, 0, 0, 0, 0, 0)$$

Hence

$$\Pi_{a \vee b} = \pi_{bc} + \pi_{ca} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, it follows from Lemma 2 that

$$S^T\Pi_{a\vee b} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The p_A satisfying (5) are the projectors onto the vectors

$$\mathbf{e}_a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_{(ab)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence

$$\Pi_{a\vee b} = \pi_a + \pi_{ab} + \pi_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \neq \pi_a + \pi_b$$

The other joins in $\mathcal{H}_V$ are built in the same way.

3. COMMUTATION RELATIONS

At first sight, the proposed representation of property lattices seems inconsistent with quantum mechanical intuition, since the operators associated with the atoms of a property lattice all commute. To reason about the commutativity of observables we must somehow take into account both the bra and the ket representation at once.

For instance, consider the pair of operators P_a and $P_{(ab)}$. They act from the bra-space to the ket space; therefore it makes no sense to speak about their commutation, since they cannot be multiplied. To speak about commutation relations, we have to render them to the same space. Note that we already have the operator doing it, namely, the sandwich matrix S of (7), and the operators SP_a and $SP_{(ab)}$ will already act in the same space, having the form:

$$SP_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \qquad SP_{(ab)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It can be checked directly that both are idempotents: $(SP_a)^2 = SP_a$ and $(SP_{(ab)})^2 = SP_{(ab)}$. We could try to make them projectors (that is, self-adjoint operators) by introducing a scalar product in the bra-space in, say, the usual Euclidean way, although, unlike the quantum mechanical situation, they will not be self adjoint: $(SP_a)^T \neq SP_a$, $(SP_{(ab)})^T = SP_{(ab)}$.

We have to introduce commutation relations in such a way that they could grasp the entire structure of the property lattice. When we try to use the above projectors P_a, P_b, etc., we immediately see that they all commute, which contradicts the violation of distributivity (1). The idea we put forth is the following. Since all the operators associated with the elements of the lattice act from $\mathcal{H}_V$ to $\mathcal{H}_\Lambda$, we can render them into one space, namely $\mathcal{H}_V$, by multiplying all of them by the matrix S from the right side. Then define the new product $\circ$ of operators in $\mathcal{H}_V$,

$$A \circ B := ASBS$$

and calculate all commutators $[P_u, P_v]$ for all $u, v \in V$.

Lemma 3. For any $u, v \in V$

$$[P_u, P_v] = S_{uv}(P_u - P_v) = \begin{cases} = 0 & \text{if } S_{uv} = 0 \\ \neq 0 & \text{otherwise} \end{cases} \tag{8}$$

Proof. The proof is obtained from direct checking by multiplication of appropriate matrices.

The results of the calculation are shown in Fig. 2, where the vertices associated with commuting projectors are linked by lines and the pairs of vertices which are not connected by a line are associated with noncommuting projectors. Now the correspondence between commutativity of projectors

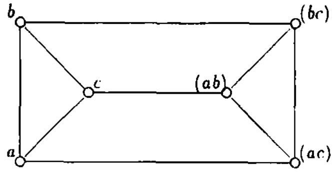

Fig. 2. Commutativity of atomic projectors.

and distributivity of the elements of the lattice is established. For instance, the atomic topologies *a*, *b*, *c* form a distributive triple, and the appropriate projectors pairwise commute.

4. CONCLUDING REMARKS

So, we see that the matrix representation of the topology lattice for three points is possible if one uses the two spaces called bra- and ket-spaces. Contrary to usual quantum mechanics, it is impossible to identify these spaces and to have the usual wave function formulation in terms of vectors in one space. It is only for the special case of the 3-point set on which all possible topologies are studied that it is possible to reduce the construction to one space, since the bra- and ket-spaces are isomorphic only when $n = 3$.

Noncommutativity of some of these matrices can lead to complementarity (as in the case of a Stern–Gerlach experiment) and to quantum jumps for the topologimeter. Nevertheless one must stress that the example of the topologimeter for the lattice of topologies for three points is an example of a totally new system, different from both classical and quantum systems. From this one comes to the conclusion that quantum topology cannot be thought of as some usual quantum system described by the wave functions as vectors in one Hilbert space, but is a new formalism for which a new interpretation is needed.

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